NRMC Probability Challenges 2020

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November 29, 2020

Problem 2020.2

You want to find someone whose birthday matches yours. What is the smallest number of strangers whose birthdays you need to ask about to have a 50-50 chance of matching?

SOLUTION

Goal
Here we would like to know how many people *you* would need to interact with such that the chances (the probability) that one of those people has the same birthday as you is 50-50 ($\frac{1}{2}$).

Solution Strategy

The problem statement ask us to find *number of people* you would need to interact with to force a specific probability. If we can derive an expression for the probability of you sharing a birthday with a stranger as a function of the number of people you intend to talk(n) to, then we can set that probability function equal to $\frac{1}{2}$ and solve for n. **Note, there are helpful hints on page 4**

Solution Details

An approach to this problem is to first figure out the probability that someone does not match your birthday and use the complement of that expression to get the probability function we care about.

Let N be the number of days in a year ($N = 365$ in this case but doesn't have to). Now imagine you run into a person, how many of the days in the year could that person be born on that are **not** your birthday? The answer is the number of days in a year minus one day (your birthday) or

$$N - 1 \text{ days.}$$

Then the probability of the person not sharing your birthday is the number of days they could be born on divided by all of the days in a year or

$$\frac{N-1}{N}.$$

Now imagine you ran into two people. Then the probability that person 1 *and* person 2 both **do not** share your birthday is:

$$\frac{N-1}{N} \times \frac{N-1}{N} = \left[\frac{N-1}{N}\right]^2.$$

And if you asked 3 people the probability would be given by $\left[\frac{N-1}{N}\right]^3$. So in general, if you spoke to n people, the probability that none of them share your birthday is

$$\left[\frac{N-1}{N}\right]^n$$

. Now that we have the probability that none of the n people you query share your birthday, we can use the first helpful hint (H1) to get the probability that at least one of the people matches your birthday.

$$P(match) = 1 - \left[\frac{N-1}{N}\right]^n.$$

So now we just need to solve the following expression for n.

$$\frac{1}{2} = 1 - \left[\frac{N-1}{N}\right]^n.$$

We can use the second helpful hint to re-write part of the expression.

$$\left[\frac{N-1}{N}\right]^n = \left[1 - \frac{1}{N}\right]^n \approx e^{-\frac{n}{N}}$$

So we need to solve

$$P(match) \approx 1 - e^{-\frac{n}{N}} \approx \frac{1}{2}$$

After taking the natural logarithm of the expression and multiplying by $-N$ we get

$$n \approx .693N.$$

Therefore for $N = 365$, you should query $n \approx 253$ people to have a 50-50 chance of one at least one of them sharing your birthday.

Helpful Hints

In order to work this problem it may be helpful to know a few things (only the bold sentences matter everything else is detail):

- **H1.If the probability of an event (E) happening is $P(E)$ then the probability of it not happening is $1 - P(E)$.**

  [*explanation*]
  The above idea follows almost immediately from the Kolmogorov axioms (rules of standard probability spaces). The axioms are listed below but firs here is some terminology you should know:

  Let (Ω, F, P) be a measure space (doesn't matter what that is) with $P(E)$ being the probability of some event E. Then we will call (Ω, F, P) a probability space, with sample space Ω, event space F and probability measure P.

  A1. The probability of an event is a positive real number.
  In math talk : $P(E) \in \mathbb{R}$ and $P(E) \geq 0, \qquad \forall E \in F$

  A2.the probability that at least one of the events in the entire sample space will occur is 1: $P(\Omega) = 1$

  A3. The probability of the union of mutually exclusive events is the sum of the probability of each individual event:
  $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$.
  Ex: The probability of getting heads(H) or tails(T) in a coin toss is $P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$. Because if you are flipping a coin you will definitely get either heads or tails. With that out of the way here is the proof of they above statement:

  Given A and A^c are mutually exclusive where A^c is the complement of A and $A \cup A^c = \Omega$:
  $P(A \cup A^c) = P(A) + P(A^c)$ ... (by A3)
  and $P(A \cup A^c) = P(\Omega) = 1$ ... (by A2)
  $\Rightarrow P(A) + P(A^c) = 1$
  $\therefore P(A^c) = 1 - P(A)$

  QED

- **H2. To first order, $1 - 1/N \approx e^{-\frac{1}{N}}$, for N sufficiently larger than 1**

This follows from the Taylor series expansion of e. I wont go into a proof of that here but, if you are curious about it, feel free to email me and we can chat about it.