Understanding Schwarz-Christoffel Transformations and Their Applications

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1 Introduction

The purpose of this research project is to analyze the construction of the Schwarz-Christoffel (S-C) transformations and understand the importance of its application to fluid flow and electric potentials. These (S-C) transformations were developed independently by Elwin Christoffel and Hermann Schwarz [Ber07, pp. 3]. The transformations themselves are a conformal mapping of the extended upper half plane $\mathbb{H}$, defined as

$$\mathbb{H} : \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},$$

to a simple polygon.

A conformal mapping or transformation by definition is an analytic function that preserves local angles (magnitude and location) [Kra12]. Since many electric potential and fluid flow problems have boundary conditions that can be modeled by a polygon, the transformations can be helpful in creating this interior. However, with most of these applications, the mapping is from a polygon to a plane, which requires the inverse Schwarz-Christoffel transformation. These inverses often cannot be found theoretically and must be done numerically.

2 Constructing the Transformation

First, the theorem will be stated in its entirety:

Theorem:*Let P be the interior of a polygon Γ having vertices $\{w_1 \dots w_n\}$ and interior angles $\{\alpha_1\pi \dots \alpha_n\pi\}$ in counterclockwise order. Let S be any conformal, one-to-one map from the upper half plane $\mathbb{H}$ onto P sat-*

isfying $S(\infty) = w_n$. *Then* S *can be written in the form:*

$$w = f(z) = A + C \int_{z_1}^{z} \prod_{i-1}^{n-1} (m - z_i)^{\alpha_i - 1} dm \qquad (1)$$

where A *and* C *are complex constants, and* $z_1 < z_2 < \cdots < z_{n-1}$ *are real numbers satisfying* $S(z_i) = w_i fori = 1, \ldots n - 1$.

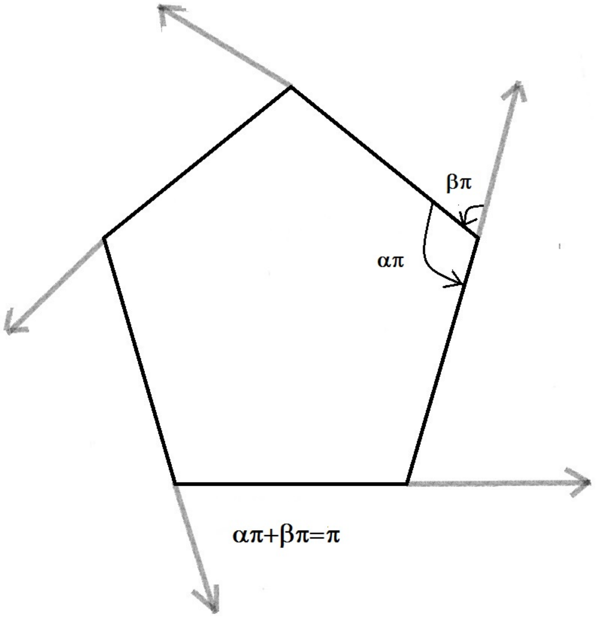
For the theorem, a polygon is defined as a simple linear curve. Thus, all polygons in this case satisfy the Jordan Curve Theorem. Moreover, P is an open simply connected domain. Then, by the Riemann Mapping Theorem, there exists a conformal map from the upper half plane to P [Ber07, pp. 7]. An important step for creating the function itself is to realize that the real axis will get mapped to the boundary of the polygon, Γ.

Now that it is clear that a mapping from $\mathbb{H}$ to a polygon exist, we will discuss each part of the expression.

Recall that Schwarz-Christoffel transformations are conformal mappings. From this information, we know that the mappings will preserve, not only the magnitude but the locations of angles that they act on. The sum of the interior angles of P, $\alpha_i\pi$, is equal to $(n-2)\pi$, or

$$\sum_{i=1}^{n} (\alpha_i) = n - 2 \qquad (1.1)$$

Equation (1) is motivated from the requirement that our polygons make a complete turn of 2π. We define the turning angle $(\beta\pi)$ at a vertex as the angle created by extending the incoming side and sweeping from the extended incoming side towards the outgoing side so that the magnitude of this sweeping motion, either clockwise or counterclockwise is less than π.



Because the sum of the turning angle and the interior angle must be π, we see that the restriction that a complete turn about our polygon be equal to 2π is really saying that $\pi(n-2)$ must equal the the sum of the interior angles e.g, π for a triangle and 2π for a square.

The construction of the equation relies on raising negative real numbers to a power. For multivalued functions, the normal branch cut is the negative real axis. For our construction, though, we will use the branch cut of the negative imaginary axis [Ber07, pp. 5].

To arrive at (1), we begin by looking at a transformation, S, that sends $\mathbb{H}$ to $\mathbb{H}$ and preserves the point at ∞. Linear fractional transformations that send $\mathbb{H}$ to $\mathbb{H}$ are one of two forms: either

$$w = c(z-a), \quad a \in \mathbb{R}, \quad c > 0$$

or

$$w = b + \frac{c}{z-a}, \quad a, b \in \mathbb{R}, \quad c < 0.[\text{Ber07}, pp.\ 6]$$

For $S(\infty) = \infty$, the transformation most be of the first form. This transformation is equivalent to

$$w = c\int_a^z dm.$$

The equation above is of the same form as the Schwarz-Christoffel transformations [Ber07, pp. 14].

Now consider a mapping, S, from $\mathbb{H}$ to the first quadrant of the w-plane. The quadrant has vertices at 0 and ∞. Moreover, at 0, the interior angle is $\frac{1}{2}$ and the interior angle is$-\frac{1}{2}$ for ∞. A transformation that does this mapping is the square root of one of the forms of the linear fractional transformation that maps $\mathbb{H}$ to $\mathbb{H}$ [Ber07, pp. 15]. The two prevertices must be $z_1 \in \mathbb{R}$ and $z_2 = \infty$ by (1). For the first case, let $S(z_1) = 0$ and $S(z_2) = \infty$. The linear fractional transformation that satisfies these conditions is of the first form. Thus,

$$S(z) = c(z - z_1)^{1/2} = \frac{1}{2}c \int_{z_1}^{z} (m - z_1)^{-1/2} dm.$$

This transformation is in the form of a Schwarz-Christoffel [Ber07, pp. 15]. For the second case, let $S(z_1) = \infty$ and $S(z_2) = 0$. These conditions are evident of the second form of the linear fractional transformation. Hence,

$$S(z) = \left(\frac{c}{z - z_1}\right)^{1/2} = -\frac{1}{2}c_1 \int_{-\infty}^{z} (m - z_1)^{-3/2} dm.$$

This equation also satisfies the equation for a Schwarz-Christoffel transformation [Ber07, pp. 16]. Additionally, special attention must be given to the singularity at z_1. Because of the singularity, if $z < z_1$, we have to integrate from $-\infty$ to z, and if $z > z_1$, we have to integrate from ∞ to z.

To arrive at our equation, consider the points, which are all on the real axis,

$$z_1 < z_2 < \cdots < z_n.$$

We need an function such that as m traverses through the points z_i, the argument of the function increases by the angle. Consider the function $f'(z)$ (independent for now from f in theorem 1) ,

$$f'(z) = (z - z_1)^{\alpha_1 - 1}(z - z_2)^{\alpha_2 - 1} \ldots (z - z_n)^{\alpha_n - 1}.$$

Thus, we can find the argument of this function:

$$\arg(f'(z)) = (\alpha_1 - 1)\arg(z - z_1) + (\alpha_2 - 1)\arg(z - z_2) + \ldots (\alpha_n - 1)\arg(z - z_n).$$

Let the point z be on the real axis. When $z < z_i$, $\arg(z - z_i) = \pi$. On the other hand, when $z > z_i$, $\arg(z - z_i) = 0$. As we go past each point z_i, the argument of our function increases by $(1 - \alpha_i)\pi$, which is equal to the turning angle at z_i. After z has gone through all of the points z_i and $z > z_n$, $\arg(f'(z)) = 0$.

Therefore, we will have reached the point we had started at. For example consider the points z_1 and z_2. Assume that $z_1 < z_2$. For $\arg(f'(z))$,

$$\arg(f'(z)) = \begin{cases} 0 & z_1 < z_2 < z \\ (\alpha_2 - 1)\pi & z_1 < z < z_2 \\ (\alpha_1 - 1)\pi + (\alpha_2 - 1)\pi & z < z_1 < z_2 \end{cases}$$

This gives a simple example of the change of the $\arg(f'(z))$ as z traverses across the prevertices z_i. This affirms our equation,

$$f'(z) = \prod_{i=1}^{n} (z - z_i)^{\alpha_i - 1} \qquad (1.2).$$

To allow for the rotation, dilation, and translation of our polygon, complex constants, A and C are used. The variable we integrate over is dummy variable. We choose to integrate over m. We now have

$$f(z) = A + C \int_{z_1}^{z} \prod_{i=1}^{n} (m - z_i)^{\alpha_i - 1} dm.$$

The equation above does not yet match the equation we are trying to find. One of the points z_i is usually chosen to be infinity. More often than not that point is z_n. Thus, the resulting information from integrating $(m - z_n)^{\alpha_i - 1}$ can be absorbed in the complex constant, C. We now have the final form of the equation:

$$f(z) = A + C \int_{z_1}^{z} \prod_{i=1}^{n-1} (m - z_i)^{\alpha_i - 1} dm.$$

Example 1 To show how the transformation works, we will look at the equation for a triangle. Let our prevertices be $z_1 < z_2 < z_3 = \infty$ and their corresponding interior angles be $\alpha_1\pi$, $\alpha_2\pi$, and $\alpha_3\pi$ [Ber07, pp. 28]. The resulting transformation is

$$w = f(z) = \int_{z_1}^{z} (m - z_1)^{\alpha_1 - 1} (m - z_2)^{\alpha_2 - 1} dm.$$

To allow for translation and dilation, constants A and C are defined such that

$$w = f(z) = A + C \int_{z_1}^{z} (m - z_1)^{\alpha_1 - 1} (m - z_2)^{\alpha_2 - 1} dm.$$

To find the equation that is not in integral form for an equilateral triangle, the beta function is normally utilized [Siu07, pp. 7].

3 Application to Electric Potentials

Conformal mapping and the Schwarz-Christoffel transformations are often used in boundary value problems for electric potentials[McD00, pp. 2]. The electric potential is defined as

$$V(\mathbf{r}) \equiv -\int_{O}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \qquad (2)$$

In this equation, $\mathbf{r}$ is just a point in space, O is a standard reference point, $\mathbf{E}$ is the electric field, and $d\mathbf{l}$ is the infinitesimal displacement vector. Also, for this to be a valid equation, the curl of the electric field must be 0, and the closed line integral of the electric field must be 0. Thus,

$$\nabla \times \mathbf{E} = 0$$

and

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0.$$

The electric field at a point charge is equal to the force applied to the point charge by all other charges in the area. To find the electric field, one must integrate over the infinitesimal point charge.Therefore,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{|\mathbf{r} - \mathbf{r'}|^2} \hat{r} dq.$$

Here $\mathbf{r}$ and $\mathbf{r'}$ are two points in space, ϵ_0 is the permittivity constant, and dq is the infinitesimal charge. The $\hat{r}$ in the function is the unit vector in the direction of distance between the two points, $\mathbf{r}$ and $\mathbf{r'}$. Thus, it can be proven using the fundamental theorem of gradients,

$$\mathbf{E} = -\nabla V,$$

where V is the potential as given in (2) [Gri12, pp. 79]. One of the most important laws in electrostatics is Gauss's Law. Gauss's Law states that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

For this equation, ρ is the volume charge density over a surface. Therefore,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}.$$

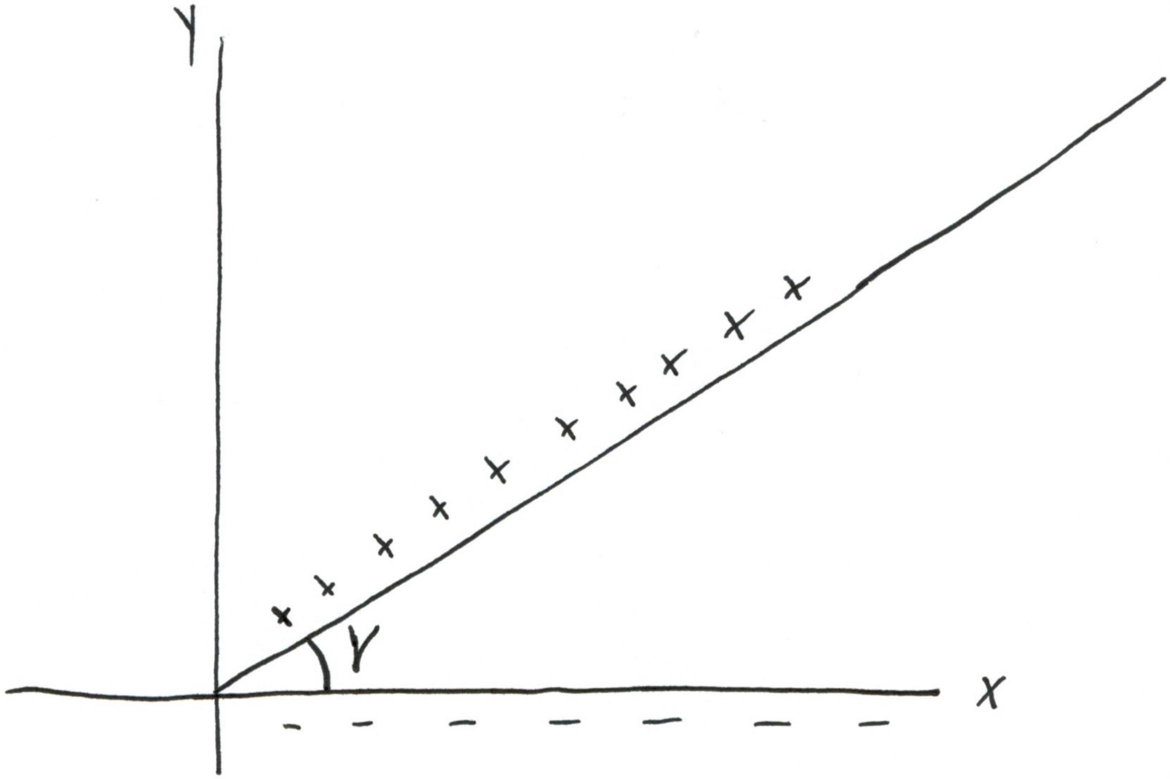
This equation is known as Poisson's equation. Most electric potential boundary problems are concerned with regions where ρ is 0. Thus, Poisson's equation becomes Laplace's equation,

$$\nabla^2 V = 0$$

A solution to Laplace's equation is unique if the boundary of the surface is given. Hence, if a potential satisfies Laplace's equation on the boundary of a surface, the potential is the same inside the surface.[Gri12]

The solutions to Laplace's equation are called harmonic functions, which must be twice continuous and differentiable [Gri12, pp. 14]. According to Kirk McDonald, all complex analytic functions gives two real valued harmonic functions [McD00, pp. 2]. Mcdonald proves this by appealing to Cauchy-Riemann equations. When solving electric potential problems, issues arise in the set up of the problem. Most of these problems start with boundaries that are polygonal, and a mapping to another plane is not necessary. Hence, instead of the original Schwarz-Christoffel transformation, the inverse of the transformation is more useful. For most polygons, the inverse of the Schwarz-Christoffel transformations can only be found numerically. If the polygon has more than 3 vertices, then without symmetry, there is no analytic solution [O'C06]. Research has been done to help find the inverses. One group, using the Jacobi Elliptic function, created an algorithm to find the prevertices of the Schwarz-Christoffel transformations[CAdM^{+}10] .

Example 2 One of the rare examples of an electric potential problem that can be solved using the Schwarz-Christoffel transformation is given here [SN10, p.333]. Consider a metal block with a wedge cut out of it. A picture of this wedge is shown on the next page.



Sometimes wedge regions such as this are used to study the diffraction of laser beams. To model this wedge, the x and y values in real space are substituted to the same values in the complex plane. For example, if a point inside the wedge was $(1, 2)$ it gets translated to $(1, 2)$, or $1 + 2i$, in the complex plane. We let the potential along the edges of the block be equal to ϕ_0. Our goal is to find the potential inside of the wedge. The wedge is cut out of the block such that from $0 \le \arg(z) \le \gamma$ the entire wedge is contained. The point of the wedge is at the origin of the complex z-plane, and the wedge extends to infinity. Using the Schwarz-Christoffel transformation of

$$w = f(z) = A + C(z - x_1)^{\alpha_1 - 1}$$

we can map the real axis of the z-plane to the wedge in the w-plane, where $w = u + iv$. To send the wedge in the w-plane to the real axis in z, we solve for z in terms of w

$$z = x_1 + (\frac{w - A}{C})^{1/\alpha - 1}.$$

Finally, to send the wedge in the z-plane onto the real axis of the w-plane, we interchange z and w

$$w = u_1 + (\frac{z - A}{C})^{1/\alpha - 1}.$$

For our wedge, u_1 and A are both 0, γ/π is equal to $\alpha - 1$, and C is equal to 1. Thus,

$$w = z^{\pi/\gamma}.$$

Since the boundary was mapped to the u-axis in the w-plane, our boundary conditions stay the same, and

$$\phi(u, 0) = \phi_0.$$

Since the potential is invariant over u, Laplace's equation becomes

$$\frac{d^2\phi}{dv^2} = 0$$

and

$$\phi(v) = cv + b.$$

Our boundary conditions gives b

$$\phi(v) = cv + \phi_0.$$

Furthermore, since $\mathbf{E} = -\nabla V$,

$$\frac{d\phi}{dv} = c = -E_0.$$

Therefore,

$$\phi(v) = \phi_0 - E_0 v.$$

Now we know that

$$v = \mathrm{Im}(w) = \mathrm{Im}(z^{\pi/\gamma}) = \mathrm{Im}((r^{\pi/\gamma})e^{i\pi\theta/\gamma}) = r^{\pi/\gamma}\sin(\pi\theta/\gamma).$$

Hence, since $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$,

$$\phi(x, y) = \phi_0 - E_0(x^2 + y^2)^{\pi/2\gamma}\sin[(\pi/\gamma)\arctan(y/x)].$$

This equation is equal to the electric potential inside of the wedge.

A more useful method to solve electric potential problems is a conformal mapping, which sends the complex plane to the complex plane. The mapping allows hard problems in one complex coordinate system to be solved more easily in another complex coordinate system. This method is similar to mapping the Cartesian coordinate system to spherical coordinates. Usually, the mapping must be found by inspection. Furthermore, one drawback to the conformal mapping method is it is only valid for problems of two variables. Nevertheless, assumptions can be made to treat three variable systems as a two variable system [Kel].

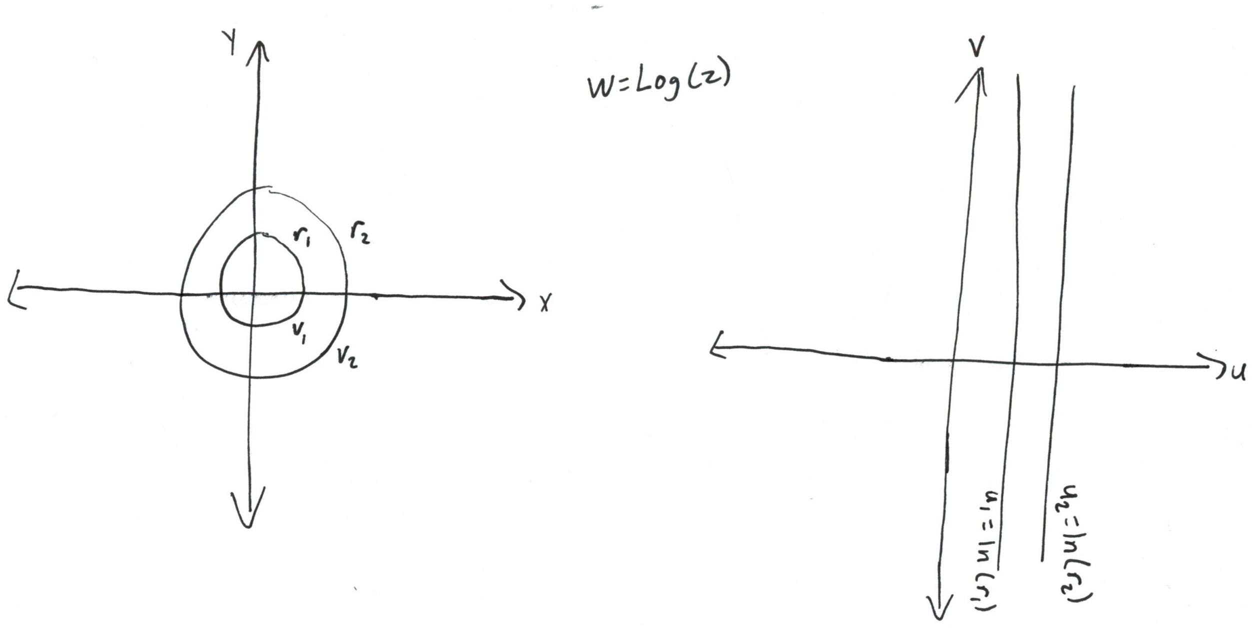
Example 3 For an example, consider two coaxial cylinders of infinite length [Kel]. Two coaxial cylinders are normally used to model a wire with an insulator in between; however, this example excludes an insulator. Thus, their cross sections would be circles in the plane. Let the first cylinder, C_1 have a radius of r_1, and let the second cylinder, C_2 have a radius of r_2. Thus,

$$x^2 + y^2 = r_1^2$$

and

$$x^2 + y^2 = r_2^2.$$

Furthermore, a potential of V_1 is placed on the surface of the first cylinder, and the second cylinder has a potential of V_2. To find the electrostatic potential between the two cylinders, each of the functions above is plotted in the complex z-plane. Then the equations are mapped to another complex plane using $w = Log(z)$. Geometrically, the mapping is sending a circle in the z-plane to a line in the w-plane. A figure of both the z-plane and w-plane is below.



The real axis for the w-plane is u, and the imaginary axis is v. For this function, $w = Log(z)$, a branch cut is defined at π. Thus,

$$z = e^w = e^{u+iv} = e^u e^{iv} = e^u(\cos(v) + i\sin(v)) = x + iy$$

$$x^2 + y^2 = e^{2u}(\cos^2(v) + \sin^2(v)) = e^{2u}$$

Therefore, $u_1 = \ln(r_1)$ and $u_2 = \ln(r_2)$. The function for the potential between the two cylinders must be of one variable, u. Since the function must obey Laplace's equation, it is of the form $mu + b$.

$$V(u_1) = m(u_1) + b = V_1$$

$$V(u_2) = m(u_2) + b = V_2$$

$$m = \frac{V_1 - b}{u_1}$$

$$V_2 = \frac{(V_1 - b)u_2}{u_1} + b$$

$$V_2 - \frac{V_1 u_2}{u_1} = b - \frac{b u_2}{u_1}$$

$$b = \frac{V_2 u_1 - V_1 u_2}{u_1 - u_2}$$

We knew the potentials at both $u = u_1$ and $u = u_2$. We were able to find our value for b by inserting them into our equation for V.

$$V_1 = m(u_1) + \frac{V_2 u_1 - V_1 u_2}{u_1 - u_2}$$

$$m = \frac{V_1 - \frac{V_2 u_1 - V_1 u_2}{u_1 - u_2}}{u_1}$$

$$m = \frac{V_1 - V_2}{u_1 - u_2} = \frac{V_2 - V_1}{u_2 - u_1}$$

With the b we just found, we were able to find the general form of V. Hence,

$$V(u) = \frac{V_2 - V_1}{u_2 - u_1} u + \frac{V_2 u_1 - V_1 u_2}{u_1 - u_2}.$$

To find our function in terms of Cartesian coordinates, we have to remember our substitutions: $u = \ln(\sqrt{x^2 + y^2})$, $u_1 = \ln(r_1)$, and $u_2 = \ln(r_2)$.

$$V(x, y) = \frac{V_2 - V_1}{\ln(r_2) - \ln(r_1)} \ln(\sqrt{x^2 + y^2}) + \frac{V_2 \ln(r_1) - V_1 \ln(r_2)}{\ln(r_1) - \ln(r_2)}$$

$$V(x, y) = \frac{V_2 - V_1}{\ln(r_2) - \ln(r_1)} \ln(\sqrt{x^2 + y^2}) + \frac{V_1 \ln(r_2) - V_2 \ln(r_1)}{\ln(r_2) - \ln(r_1)}$$

This function gives the potential between two coaxial cylinders of infinite length.

For both examples, Laplace's equation was useful in simplifying the method to find the potentials. However, there are not many other similarities between the examples. The cylinder example had a predetermined conformal mapping while the wedge example required the Schwarz-Christoffel transformations. The wedge example, though, yielded the equation for the potential more easily than the cylinder example. Most boundary value electric potential problems do not require the electric field to determine the potential, but the first example required it.

4 Applications to Fluid Mechanics

In ideal fluid flows, it is assumed that fluid, when coming in contact with a surface, makes a sharp turn around the boundary and recombines without ever separating. This assumption results in difficulties when trying to find the velocity about certain areas (such as corners) on boundary. This difficulty motivated the theory of free-streamline. The free stream line theory uses Schwarz-Christoffel transformations to obtain the mapping of a polygon (representing a surface obstructing flow) to $\mathbb{H}$ in order to find an expression for the velocity of the fluid. It should be noted, however, that the theory of Free Streamlines still treats fluids as ideal. respects.[Cha09, pp. 19]

For our purposes we will consider fluids that are planar (2D) and steady state. Steady state here means that the properties of the fluid (at any one point) are time invariant.

Consider a fluid whose velocity vector field is given by :

$$\mathbf{v}(\mathbf{x}) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \text{ at the point } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D} \;\; [\text{Olv15}, pp.\ 19]$$

$\mathbf{x}$ in $\mathcal{D} \subset \mathbb{R}^2$ is the location in space that the fluid is occupying while $\mathbf{v}(\mathbf{x})$ is the instantaneous of the fluid velocity at that point. We consider ideal fluids to be frictionless, incompressible and irrotational. A fluid is incompressible if its volume does not change when it is flowing. From relations between conservation and flux of mass and the divergence theorem, we can express incompressibility of a fluid mathematically as its velocity having a zero divergence:

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (4.1).$$

The combination of the no friction and incompressibility constraints placed on an ideal fluid results in it being irrotational as well. If we consider the forces acting on a single parcel of fluid (a perfect sphere),these surface forces are modeled to act on the center of the sphere. Modeling the forces in this way means there can be no torque on the fluid parcel which implies a zero angular momentum, ensuring the fluid will have no rotation. Irrotational fluids are expressed by zero valued curls[Cha09, pp. 19]:

$$\nabla \times \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (4.2).$$

We can reweite equations (4.1) and (4.2) as:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}\ , \qquad \frac{\partial u}{\partial y} = +\frac{\partial v}{\partial x}$$

It is because the requirements of an ideal fluid look like the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(except for the negative sign in on the ∂v terms) that we can describe fluid flow with complex functions,

$$f(z) = u(x, y) - iv(x, y)$$

where $f(z)$ must be a complex analytic function of $z = x + iy$. We know that a complex flow must have this form from Theorem 4.5 in [PJO,19] Consider the complex anti derivative of $f(z)$ we can get a function

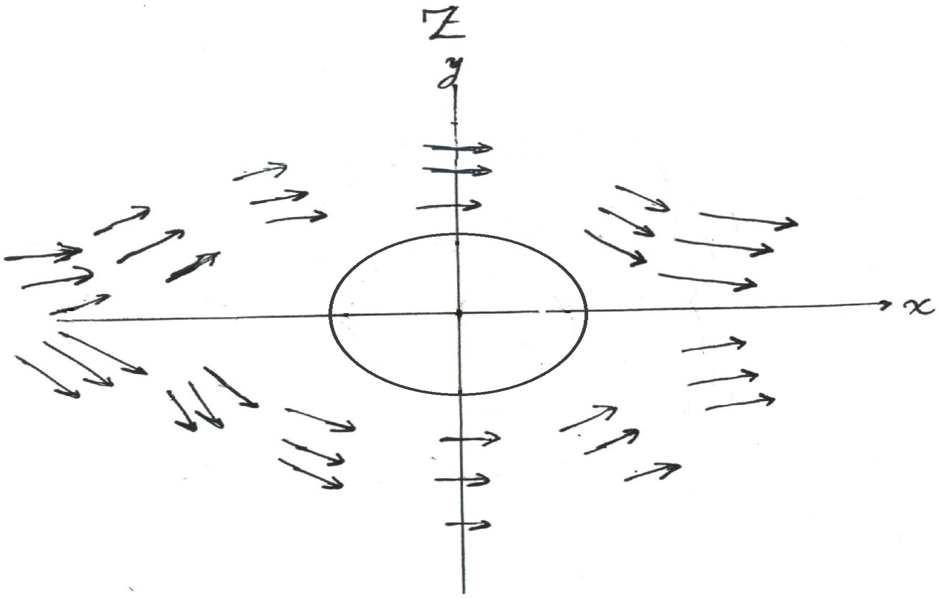
$$\eta(z) = \phi(x, y) + i\Psi(x, y)[\text{Kra12}, pp\ 254]$$

where

$$f(z) = \frac{d\eta}{dz} \ (\forall z \in \mathcal{D})$$

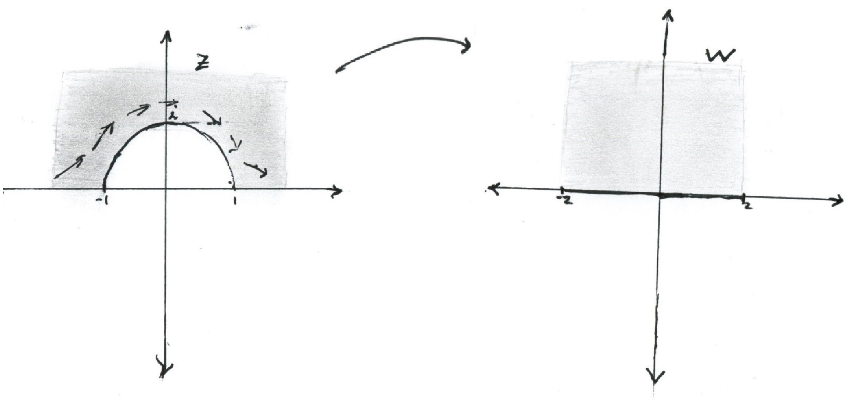
known as the velocity potential or the complex potential function for the fluid flow. As an example of concept, we will use a conformal mapping to find the velocity.

Example 4 Consider a fluid flowing around a circular post of radius 1 centered at the origin(figure below)[Kra12, pp. 254].We will assume that the fluid is flowing from the negative x direction to the positive x direction in the complex plane (that is, its velocity will have the form $f(z) = u(x, y) - iv(x, y)$).



We want to get an expression for the velocity flowing around the post. By symmetry we can focus our efforts on the upper half of the plane that contains the upper half of the circular post.

We will map the boundary of our first region z-plane to the boundary of the w-plane. This means that we want to use a conformal mapping that will take the upper half of the complex unit circle to the w-plane and we have chosen a mapping specifically to the real line in w:



$$w = f(z) = z + \frac{1}{z}.$$

Notice that f takes

$$-1 \to -2$$

$$i \to 0$$

$$1 \to 2.$$

Now if we are given a complex potential function for flow in the w-plane say,

$$G(w) = Aw \text{ (where A is a constant)}$$

we can find the corresponding potential (η) in z by composing G with f.:

$$\eta(z) = G\big(f(z)\big) = A \cdot \left(z + \frac{1}{z}\right).$$

Now we can use the Cauchy Riemann equations to notice that the velocity is the derivative of the complex potential function:

$$\eta'(z) = \frac{\partial \phi}{\partial x}(x, y) + i\frac{\partial \psi}{\partial x}(x, y).$$

Where, because $\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$, we can write

$$\eta'(z) = \frac{\partial \phi}{\partial x}(x, y) - i\frac{\partial \phi}{\partial y}(x, y).$$

Because of how we defined η, we have that the velocity $\mathbf{V}$ is given by:

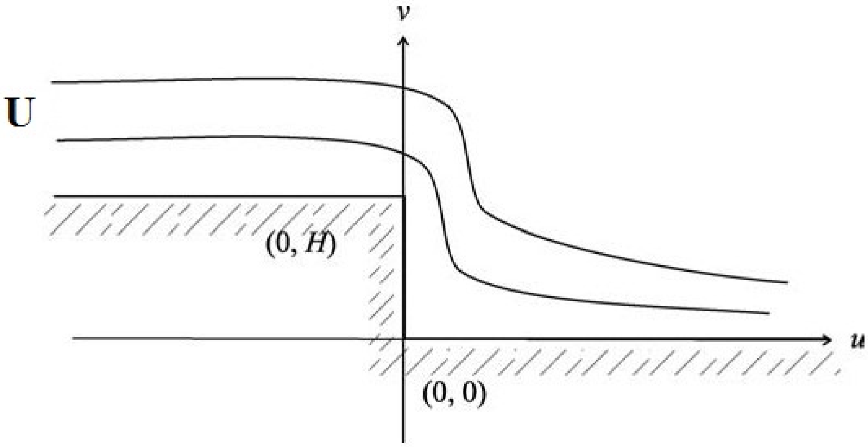
$$V = \overline{\eta'(z)}$$

$$v = A \cdot \left(1 - \frac{1}{z^2}\right).$$

We can get information about the velocity around the post in the z-plane : as the fluid moves in any direction away from the post the velocity tends to a constant in direction and magnitude. In other words the flow is uniform before and after interfering with the post.

So we see that in principle given a post of any shape we would solve a inverse Schwarz-Christoffel problem to get the conformal mapping from the polygon P to the upper half plane and use this mapping to derive physical quantities.

Example 5. Consider the potential flow over a vertical step of height H, the configuration of which is shown in the Figure below. The flow upstream far from the step is uniform with constant speed U and parallel to the floor bottom. We want to find the speed along the bottom edges of the flow field.[Kwo10, pp. 404]



In this example our polygon is the step boundary that the fluid is flowing down from. In order to solve this problem we need to construct the mapping $w = f(z)$ that would take the points $(x_1 \cdots x_n)$ from the z-plane to the w-plane where our Polygon exist. Our polygon has three vertices: $(0, 0)$,$(0, H)$ and a virtual vertex at infinity. We will assign values to our x points: $x_1 = -1$, $x_2 = 1$ and $x_3 = \infty$. The corresponding points we are mapping to are $w_1 = iH$, $w_2 = 0$ and $w_3 = \infty$.We can tell that our interior angles α_k are $\frac{\pi}{2}$,$\frac{-\pi}{2}$ and 0 at infinity. Using Eq(1.4) we see that

$$f'(z) = A(z + 1)^{\frac{1}{2}}(z - 1)^{-\frac{1}{2}}$$

Where A is a constant that can be found using the relation in [Kwo10, pp. 401].

$$A = \frac{|w_2 - w_1|}{\left| \int_{x_1}^{x_2} \prod (m - x_k)^{\alpha_k - 1} dm \right|}$$

where $|w_2 - w_1| = H$. Working the integral in the bottom leaves A to be:

$$A = \frac{H}{\pi}$$

So we find f' to be :

$$f'(z) = \frac{H}{\pi}(z+1)^{\frac{1}{2}}(z-1)^{-\frac{1}{2}}$$

so we now can integrate to find our the transformation f that will map each of the vertices in the z plane to the vertices of the step domain in the w plane[Kwo10, pp. 404].

$$w = f(z) = \frac{H}{\pi}\left[(z^2-1)^{\frac{1}{2}} + \cosh^{-1}(z)\right].$$

With the transformation in hand we can consider the Complex potential function for the w plane :

$$\eta(w)$$

and the corresponding potential function for the z plane

$$\eta(w(z)).$$

We can relate the two potentials by the chain rule [Kwo10, pp. 405] :

$$\frac{d\eta(w)}{dw} = \frac{d\eta(z)}{dz}\frac{dz}{dw} \quad (4.3)$$

We know from our previous example that the complex potential function is related to the velocity function $v(w)$ by $\eta'(w) = v(w)$. So equation (4.3) is really a relation of complex velocities. Furthermore by the assumptions of the problem we know that far away from the step (as $w \to \infty$) v should become a constant

U. Notice though that as $z \to \infty$ $\frac{dw}{dz} = f'(z) \to \frac{H}{\pi}$ so using equation (4.3) we see that

$$\lim_{z\to\infty} \frac{d\eta(z)}{dz} = U\frac{H}{\pi} \quad (4.4).$$

Notice that $\frac{dw}{dz} = \frac{1}{\frac{dz}{dw}}$ so then that

$$\frac{dz}{dw} = \frac{1}{f'(z)} = \frac{\pi}{H}\Big(\frac{z-1}{z+1}\Big)^{1/2} \quad (4.5).$$

It follows from the combination of (4.4) and (4.5) that the equation of our velocity in the w plane is:

$$v(w) = U\Big(\frac{z-1}{z+1}\Big)^{1/2}$$

Yue tells us that the The points along the bottom edges and the vertical step of the flow field in the w-plane are the image points of $z = x$ along the real axis in the z-plane. Which implies that that our speed in terms of x can be written as

$$|v| = U\sqrt{\left|\frac{x-1}{x+1}\right|} \quad [\text{Kwo10}, \textit{pp.}\ 405]$$

So we see that the conners of our boundary ($x_1 = -1, x_2 = 1$) the speed becomes infinite.

5 Conclusion

The Schwarz-Christoffel transformations in general transform a problem in the complex plane z to the complex plane w. As a result we are able to transform functions that exist in one plane to expressions in another plane. This characteristic of the transformation becomes especially useful for problems in potential theory where electric current is modeled as a flow and in fluid mechanics where we deal with liquid flows. The remarkable applicability of the Schwarz-Christoffel transformation to flow (including Brownian motion!) may be because wherever there is flow there must be a boundary, expressed as a polygonal shape, that contains the flow.

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